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NEIGHBORHOOD FIXED PENDANT VERTICES

BY

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ABSTRACT. If x is pendant in G, then x^* denotes the unique vertex of G adjacent to x. Such an x is said to be *neighborhood-fixed* whenever x^* is fixed by $A(G - x)$. It is shown that if G is not a tree and has a pendant vertex, but no $*$-fixed pendant vertex, then there is a subgraph $G^\#$ of G such that for some $y \in V(G^\#)$, $O(A(G^\#)_y) > t!$ where t is the maximum number of edges in a tree rooted in $G^\#$.

Let G denote a finite connected graph without loops or multiple edges. Let $V(G)$, $E(G)$ and $A(G)$ denote respectively the vertex set, the edge set, and the automorphism group of G. Let $x \in V(G)$. The valency of x in G is denoted by $\mathrm{val}(G, x)$, and x is defined to be pendant in G if $\mathrm{val}(G, x) = 1$. The subgraph of G obtained by deleting x and all edges incident with x is denoted by $G - x$. If x is pendant in G, then x^* denotes the unique vertex of G adjacent to x. Such an x is said to be *neighborhood-fixed* whenever x^* is fixed by $A(G - x)$. Neighborhood-fixed will be denoted by $*$-fixed for the remainder of the paper.

1. Tree growth number. A *pruning* of G is a decomposition:

$$G = G^\# \cup T_1 \cup T_2 \cup \cdots \cup T_k,$$

where $G^\#$ is the maximal subgraph of G, each vertex of which has valency ≥ 2, and where $\{T_i\colon 1 \leq i \leq k\}$ is a nonempty set of disjoint nontrivial rooted trees each having only its root vertex x_i in $G^\#$. Note that G has a pruning if and only if it is not a tree and has a pendant vertex. Moreover, $G^\#$ is connected because G is connected, and the decomposition is unique up to the order of the T_i.

In [1], Robertson and Zimmer conjectured: *If G is a finite graph having a pruning such that*

$$\max\{|E(T_i)|\colon 1 \leq i \leq k\} > \max\{\mathrm{val}(G^\#, x)\colon x \in V(G^\#)\}$$

then G has a $*$-fixed pendant vertex.

There are counterexamples to the conjecture. One class of counterexamples is found among G with $G^\#$ a rooted tree with triangles affixed to the pendant vertices.

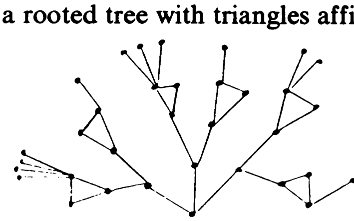

FIGURE 1

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Another counterexample can be constructed from Tutte's 8-cage, mentioned in [2]. Let $G^\#$ be the 8-cage. There exists a subset $\{x_0, x_1, x_2, x_3, x_4\}$ of $V(G^\#)$ on which $A(G^\#)$ acts as the symmetric group S_5. Then a pruning $G = G^\# \cup T_1 \cup T_2 \cup T_3 \cup T_4$, where T_i is an i-star with center the root x_i for $1 \leq i \leq 4$, defines a graph G with no $*$-fixed pendant vertex.

Let A be a finite group acting on a finite set X. Let $Y \subseteq X$ and $x \in X$. Denote the order of A by $o(A)$, the subgroup of A fixing x by A_x, and the subgroup of A fixing Y setwise by $A_{(Y)}$. The restriction of $A_{(Y)}$ to Y is denoted by A^Y. Denote the group generated by elements $\varphi_1, \varphi_2, \ldots, \varphi_k$ by $\langle \varphi_1, \ldots, \varphi_k \rangle$ and the group generated by subgroups $H_1, \ldots, H_k$ by $\langle H_1, \ldots, H_k \rangle$.

An ordered partition $X = \bigcup_{i=0}^{s} X_i$ is *reducible* if $s \geq 1$, $X_s \neq \varnothing$, and for every $x \in X_i$, $i \geq 1$, there exists a *reducing* map $\alpha \in A$ such that $(x)\alpha \in X_{i-1}$ and α stabilizes the blocks of the partition $X_0 \cup \cdots \cup (X_{i-1} \cup x) \cup (X_i - x) \cup \cdots \cup X_s$. Let $t(A, X)$ denote the largest s such that a reducible partition exists.

Note that for $x \in X_i$, $i \geq 1$, there may be many maps that reduce x. Let α_x be one such reducing map. If $Z \subset X_i$, $i \geq 1$, let $I(Z) = \{(z)\alpha_z \colon z \in Z\}$. If H is a subgroup of A that contains the group generated by the reducing maps $\{\alpha_y \colon y \in Y\}$ then H *reduces* Y. An orbit O of a subgroup H of A is a *reducing orbit* if there exists a $z \in O$ such that $\alpha_z \in H$.

Let G be a graph with a pruning. Suppose that G has no $*$-fixed pendant vertex. Let T_i, $1 \leq i \leq k$, be the trees of the pruning of G. Replace each tree T_i by a star S_i with $|E(S_i)| = |E(T_i)|$ rooted at its center x_i to form a new graph G' with a pruning. Then G' also has no $*$-fixed pendant vertex. To see this, suppose on the contrary that $x \in V(S_i)$ is a $*$-fixed pendant vertex. Then x_i is fixed by $A(G' - x_i)$. Let the weight, $w(A)$, of $A \in E(T_i)$ be the number of edges in the component of $(T_i)'_A$ (formed by deleting A from T_i), which does not contain x_i. Suppose $(x_i, A_1, a_1, A_2, a_2, \ldots, a_{k-1}, A_k, x)$ is a path in T_i from x_i to a pendant vertex x following successive lightest possible edges. Then x is a $*$-fixed pendant vertex of G, contrary to assumptions.

G will be assumed to have a pruning such that $G^\#$ is vertex-transitive and such that T_i, $1 \leq i \leq k$, is a star rooted at its center x_i. If G has a pruning and no $*$-fixed pendant vertex, the star roots induce a reducible partition of $V(G^\#)$ under $A(G^\#)$ by the rule: $X_j = \{x \colon x \text{ is the root of a } j\text{-star}\}$. Conversely, any reducible partition of $V(G^\#)$ leads to a graph G with a pruning which induces the given reducible partition. Because of this equivalence $t(A(G^\#), V(G^\#))$ is termed the *tree growth number* of $G^\#$.

2. Lemmas on permutation groups.

LEMMA 1. *Let A be a group acting on a set X with a reducible partition $X = \bigcup_{i=0}^{s} X_i$. Let $Z \subseteq X_i$, for some $i \geq 1$, and let H be a subgroup such that either* (1) *$H \subset A_{(X_i')}$ where $Z \subseteq X_i' \subseteq X_i$ and $(z)\alpha_{z'} \in X_i'$ for all distinct $z, z' \in Z$, or* (2) *$H \subset A_{(X_{i-1}')}$ where $I(Z) \subseteq X_{i-1}' \subseteq X_{i-1}$. If $K = \langle H, \{\alpha_z \colon z \in Z\} \rangle$ then $o(K) \geq$* $(|Z| + 1)o(H)$.

PROOF. Note that if $o(H) \leq 1$ or $|Z| \leq 1$ the result is clear. Let $\psi, \varphi \in H$ and $z, z' \in Z$, where $\psi \neq \varphi$ and $z \neq z'$. Then $\alpha_z\psi \neq \alpha_z\varphi$, $\alpha_z\psi \neq \varphi$, and $\alpha_z\psi \neq \alpha_{z'}\psi$. Suppose $\alpha_z\psi = \alpha_{z'}\varphi$. Suppose $H \subset A_{(X_i')}$. Then $(z)\alpha_z\psi \notin X_i'$, but $(z)\alpha_{z'}\varphi \in X_i'$. Suppose $H \subset A_{(X_{i-1}')}$. Then $(z)\alpha_z\psi \in X_{i-1}'$, but $(z)\alpha_{z'}\varphi \notin X_{i-1}'$. Hence $o(K) \geq (|Z| + 1)o(H)$.

LEMMA 2. *Let $\{n_i\}_{i=1}^s$ be a sequence of positive integers with $s \geq 2$. If $\sum_{i=1}^s n_i = N$ then $\prod_{i=1}^s(n_i + 1) \geq 2N$.*

PROOF. By induction on s.

LEMMA 3. *Let A be a group acting on a set X with a reducible partition $X = \bigcup_{j=0}^s X_j$. Suppose $Z \subseteq X_j$ for $1 \leq j \leq t$ and let $H = \langle \alpha_z\colon z \in Z\rangle$ and for every reducing orbit O of H $O \cap X_j \subset Z$. Then either $O(H) = 2^a = |Z| + 1$ for some positive integer a or $O(H) \geq 2|Z|$.*

PROOF. Let H have $k \geq 2$ orbits $O_1, \ldots, O_k$ such that $Z_i = O_i \cap Z \neq \varnothing$. Let $H_1 = \langle \alpha_z\colon z \in Z_1\rangle$. Clearly, $o(H_1) \geq |Z_1| + 1$. Since $H_1 \subset H_{(Z_2)}$ by Lemma 1 (L1), $o(H_2) \geq (|Z_2| + 1)o(H_1)$ where $H_2 = \langle \alpha_z\colon z \in Z_1 \cup Z_2\rangle$. By the repeated application of (L1), $o(H) \geq \prod_{i=1}^k(|Z_i| + 1)$. By (L2), $o(H) \geq 2|Z|$.

Let O be the only orbit of H such that $Z \cap O \neq \varnothing$. Let $|$ denote divides. Now $|O|\,|o(H)$. If $|O| < o(H)$ then $o(H) \geq 2|O| > 2|Z|$. If $|O| = o(H)$ then H is regular on O. Suppose $|O| = |Z| + 1$. Then $(z)\alpha_z^2 = z$ and by regularity α_z has order 2, thus by Cayley's Theorem $o(H) = 2^a$. Suppose $|O| > |Z| + 1$. Let $X_{j-1}' = O \cap X_{j-1}$. A reducing map sends only one element of X_{j-1}' into Z. The remaining images of X_{j-1}' lie in X_{j-1}'. Since H is regular on O, $|\{(p, q) | p \in X_{j-1}'$ and $q = (p)\psi \in X_{j-1}'$, where $\psi \in H\}| = |X_{j-1}'|^2$. Each α_z induces $|X_{j-1}'| - 1$ of these pairs. The identity map induces $|X_{j-1}'|$ pairs. Hence $(|X_{j-1}'| - 1)|Z| \leq |X_{j-1}'|^2 - |X_{j-1}'|$ so that $|Z| \leq |X_{j-1}'|$. Hence $o(H) = |O| \geq |X_{j-1}'| + |Z| \geq 2|Z|$.

COROLLARY 3.1. *Let A be a group acting on a set X. Let $s \geq 2$ and let $\bigcup_{i=0}^s X_i$ be a reducible partition of X. Let $H \subseteq A$ and suppose H has an orbit O such that $O \cap X_j \neq \varnothing$ and $O \cap X_{j+1} \neq \varnothing$, for some $1 \leq j < s$. Suppose $\langle \alpha_z\colon z \in O \cap (X_j \cup X_{j+1})\rangle \subset H$. Then H is not regular on O.*

PROOF. Let $W = O - (X_j \cup X_{j+1})$. Suppose H is regular on O. Since H is regular, each point $x \in O$ is the image of $y \in O$ by a single map of H. Reducing $X_j \cap O$ and $X_{j+1} \cap O$ requires $|(X_j \cup X_{j+1}) \cap O|$ distinct nonidentity maps of H by definition. Let $v \in X_{j+1}$. No reducing map α_x for $x \in O \cap (X_j \cup X_{j+1})$ is such that $(w)\alpha_x = v$ where $w \in W$. Hence there are $|W|$ such maps in H. However, the total number of nonidentity maps in H is only $|O| - 1$, a contradiction. Hence H is not regular on O.

COROLLARY 3.2. *Let A be a group acting on a set X. Let $X = \bigcup_{i=0}^s X_i$ be a reducible partition. Let $H = \langle \alpha_z\colon z \in X_k\rangle$ for some $k > 0$.*

(1) *Suppose H has a single reducing orbit O. Then*

(a) $o(H) = |X_k| + 1$ *when* $|X_k| = 2^a - 1$ *for* $a \geq 1$.

(b) $o(H) = 2|X_k|$ *when* $|O \cap X_{k-1}| = |X_k|$, *and*

(c) $o(H) \geqslant 2(|X_k| + 1)$ *otherwise.*

(2) *Suppose H has exactly two reducing orbits O_1 and O_2. Then*

(a) $o(H) > 2|X_k|$, *or*

(b) $o(H) = 2|X_k|$, $|O_1| = 2$, *and* $|O_2| = |X_k| = 2^b$ *for* $b \geqslant 1$.

(3) *Suppose H acts on $l \geqslant 3$ reducing orbits. Then*

(a) $o(H) \geqslant 3|X_k|$ *if* $|X_k| > p$, *and*

(b) $o(H) \geqslant 2^p$ *if* $|X_k| = p$.

Proof. Suppose H has a single reducing orbit O. If H does not act regularly on O then $o(H_x) \geqslant 2$ for $x \in O$. Since O contains a point of X_{k-1}, $o(H) \geqslant 2(|X_k| + 1)$. Suppose H acts regularly on O. By (L3), $o(H) = |X_k| + 1$ only if $|X_k| = 2^a - 1$, for $a \geqslant 1$. By (L3), $|X_{k-1} \cap O| \geqslant |X_k|$ if $|X_{k-1} \cap O| \geqslant 2$. Hence if $o(H) = 2|X_k|$ then $|X_{k-1} \cap O| = |X_k|$. Suppose H acts on two reducing orbits O_1 and O_2. Let $|O_1 \cap X_k| = b$ and $|O_2 \cap X_k| = a$ and suppose $a \geqslant b$. By (L1), $o(H) \geqslant (a + 1)(b + 1) \geqslant ab + a + b + 1$. If $b \geqslant 2$ then $o(H) > 2|X_k|$. If $b = 1$ then $o(H) \geqslant 2(a + b)$. Hence, if $o(H_1) = 2|X_k|$ then $|O_1 \cap X_k| = 1$. The unique point of $O_1 \cap X_k$ is fixed by $H' = \langle \alpha_z : z \in O_2 \cap X_k \rangle$. Since $o(H) \leqslant o(H')|O_1|$ it follows that $|O_1| = 2$. Since $o(H') = |O_2 \cap X_k| + 1$ it follows by (L3) that $|X_k| = 2^c$ for $c \geqslant 1$.

Suppose H acts on $p \geqslant 3$ orbits $O_1, \dots, O_p$. By (L1), $o(H) \geqslant \prod_{j=1}^{p}(a_j + 1)$ where $a_j = |X_k \cap O_j|$ for $1 \leqslant j \leqslant p$. Without loss of generality, $\{a_j\}_{j=1}^{p}$ is ordered so that $a_j \geqslant a_{j-1}$ for $2 \leqslant j \leqslant p$. Suppose $a_2 \geqslant 2$. Then

$$\begin{aligned}(a_p + 1)(a_{p-1} + 1) \cdots (a_1 + 1) \\ &\geqslant a_p + \cdots + a_1 + a_p(a_{p-1} + \cdots + a_1) + a_p(a_{p-1}a_{p-2} + \cdots + a_2a_1) \\ &\geqslant 3|X_k| \quad \text{if } p \geqslant 4.\end{aligned}$$

If $p = 3$ and $a_2 \geqslant 2$ then

$$\begin{aligned}(a_3 + 1)(a_2 + 1)(a_1 + 1) &\geqslant a_2a_1 + a_1 + a_2 + 1 + a_3a_1 + a_3a_2 + a_3 \\ &\geqslant (a_1 + a_2 + a_3) + (a_2a_1 + a_3a_2 + a_3a_1a_2) \geqslant 3|X_k|.\end{aligned}$$

If $a_2 = 1$ then $(a_3 + 1)(a_2 + 1)(a_1 + 1) = (a_3 + 1)4 \geqslant 3a_3 + a_3 + 4 \geqslant 3|X_k|$.

If $a_p = 1$ then $|X_k| = p$ and $o(H_1) \geqslant 2^p$ by (L1).

Lemma 4. *Let A be a group acting faithfully and transitively on a set X. Suppose $X = X_0 \cup X_1 \cup X_2$ is a reducible partition. Then neither $A_{(X_2)}$ nor $A_{(X_0)}$ contains a nontrivial normal subgroup of A.*

Proof. Let N be a nontrivial normal subgroup of A. The orbits of N are blocks of A. Since A is faithful these orbits are not singletons. Let $x_2 \in X_2$ and let $(x_2)\alpha_{x_2} = x_1$. Suppose $N \subset A_{(X_2)}$ so there exists $\psi \in N$ with a cycle $(x_2x_2' \cdots)$, $x_2' \in X_2$. Then $(x_2')\alpha_{x_2} \in X_2$ by definition, and $\alpha_{x_2}^{-1}\psi\alpha_{x_2}$ has a cycle $(x_1(x_2')\alpha_{x_2} \cdots)$. However, $\alpha_{x_2}^{-1}\psi\alpha_{x_2} \in N$ while $\alpha_{x_2}^{-1}\psi\alpha_{x_2} \notin A_{(X_2)}$, a contradiction. Suppose $N \subset A_{(X_0)}$. Let $x_1 \in X_1$ and $(x_1)\alpha_{x_1} = x_0$. There exists a $\psi \in N$ with a cycle $(x_1x_1' \cdots)$ with $x_1' \in X_1 \cup X_2$. Then $(x_1')\alpha_{x_1} \in X_1 \cup X_2$ and $\alpha_{x_1}^{-1}\psi\alpha_{x_1}$ has a cycle $(x_0(x_1')\alpha_{x_1} \cdots)$. However, $\alpha_{x_1}^{-1}\psi\alpha_{x_1} \in N$ and $\alpha_{x_1}^{-1}\psi\alpha_{x_1} \notin A_{(X_0)}$, a contradiction.

REMARK. Actually, the hypothesis that $X_0 \cup X_1 \cup X_2$ is reducible can be weakened to the existence of some reducing maps α_{x_2} and α_{x_1}.

COROLLARY 4.1. *Let* $H = \langle \alpha_z \colon z \in X_2 \rangle$ *and let* $K = \langle \alpha_z \colon z \in X_1 \rangle$. *Then* $o(A) | [A : K]!$ *and* $o(A) | [A : H]!$.

PROOF. By (L4), K and H cannot contain normal subgroups of A. However, by the core theorem of algebra (Herstein, *Topics in algebra*, Blaisdell, Waltham, Mass., 1964, p. 62), $o(A) | [A : K]!$ and $o(A) | [A : H]!$.

COROLLARY 4.2. *Let A act on a set X. Let* $X = \bigcup_{i=0}^{t} X_i$ *be a reducible partition. Suppose* $2 \leq j \leq t$. *Let* $H_1 = \langle \alpha_z \colon z \in X_j \rangle$ *and* $H_2 = \langle \alpha_z \colon z \in X_j \cup X_{j-1} \rangle$. *Suppose* $O_1, \ldots, O_k$ *are the reducing orbits of* H_2 *and* N_2 *is the pointwise stabilizer of* $\bigcup_{v=1}^{k} O_v$. *Let* $\bar{H}_2 = H_2 / N_2$. *Let* $H_1' = \langle H_1, N_2 \rangle$ *and* $\bar{H}_1' = H_1' / N_2$. *Then* $\bar{H}_1'$ *contains no normal nontrivial subgroup of* $\bar{H}_2$.

PROOF. Suppose N is a normal nontrivial subgroup of $\bar{H}_2$ contained in $\bar{H}_1'$. Then N acts nontrivially on an orbit O_s for $1 \leq s \leq k$. Also $O_s \cap X_{j-1} \neq \varnothing$, since O_s is a reducing orbit. Let $x_{j-1} \in O_s \cap X_{j-1}$. Since the orbits of N are blocks of $\bar{H}_2$ there exists a $g \in N$ containing a nontrivial cycle $(x_{j-1} x_{j-1}' \cdots)$ where $x_{j-1}' \in X_{j-1} \cup X_j$. Then $(x_{j-1})\alpha_{x_{j-1}} \in X_{j-2}$ and $(x_{j-1}')\alpha_{x_{j-1}} \in X_{j-1} \cup X_j$ by definition. Since $\alpha_{x_{j-1}}^{-1} \cdot g \alpha_{x_{j-1}}$ contains the cycle $((x_{j-1})\alpha_{x_{j-1}} (x_{j-1}')\alpha_{x_{j-1}} \cdots)$ a contradiction follows from $N \subset \bar{H}_1' \subset \bar{H}_{2(x_{j-2})}$. Hence, $\bar{H}_1'$ contains no normal subgroup of $\bar{H}_2$. As in (L4 C1) this means $o(\bar{H}_2) | [\bar{H}_2 : \bar{H}_1']!$.

LEMMA 5. *Let A be a group acting on a set X. Let* $X = \bigcup_{i=0}^{s} X_i$ *be a reducible partition. Suppose* $1 \leq j \leq s$ *is such that* $|X_j| \geq 2$. *Suppose H is a subgroup of A that stabilizes* X_j. *Let* $K = \langle H, \{\alpha_z \colon z \in X_j\} \rangle$. *Then there exists* $x \in X_j$ *such that* $o(K_x) \geq 2 o(H) / |X_j|$.

PROOF. Suppose X_j is not an orbit of H. There exists an orbit O of H such that $O \subseteq X_j$ and $|O| \leq |X_j|/2$. Hence $o(K_x) \geq o(H_x) \geq 2o(H)/|X_j|$ for any $x \in O$. Suppose X_j is an orbit of H. Let x, x' be distinct elements of X_j. Since X_j is an orbit of H there are $o(H)/|X_j|$ maps of H sending x' to $(x')\alpha_x$. Let ψ be such a map. Then $(x')\alpha_x \psi^{-1} = x'$ and also $\alpha_x \psi^{-1} \notin H_{x'}$. Hence $o(K_{x'}) \geq 2o(H_{x'}) \geq 2o(H)/|X_j|$.

Let A act on a set X. Let $X = \bigcup_{i=0}^{t} X_i$ be a reducible partition. Suppose $1 \leq j \leq t$ and $t \geq 2$. Let

$$H_l = \langle \alpha_z \colon z \in X_j \cup \cdots \cup X_{j-(l-1)} \rangle \quad \text{if } 1 \leq l \leq j$$

and

$$H_l = \langle \alpha_z \colon z \in X_1 \cup \cdots \cup X_l \rangle \quad \text{if } j < l \leq t. \tag{2.1}$$

Let O_t be a reducing orbit of H_t that intersects X_t. Let O_{t-1} be a reducing orbit of H_{t-1} contained within O_t which intersects X_{t-1} if $j < t$ and intersects X_t if $j = t$. Let $O_{t-i} \subseteq O_{t-(i-1)}$ be a reducing orbit of H_{t-i} for $2 \leq i \leq t-1$, such that O_{t-i} intersects X_{t-i} if $j \leq t-i$ and intersects X_j if $t-i < j$. For $2 \leq l \leq t$ define N_l to be the subgroup of H_l that stabilizes O_l pointwise. Let

$$\begin{aligned} \overline{H}_l &= H_l/N_l \quad \text{if } 1 \leq l \leq t-1, \\ H'_l &= \langle H_l, N_{l+1}\rangle \quad \text{if } 2 \leq l \leq t, \text{ and} \\ \overline{H}'_l &= H'_l/N_{l+1} \quad \text{if } 2 \leq l \leq t. \end{aligned} \tag{2.2}$$

LEMMA 6. *Let A act on a set X. Let $X = \bigcup_{i=0}^{t} X_i$ be a reducible partition. Suppose $1 \leq j \leq t$ and $t \geq 2$. For $2 < l \leq t$, u_l is defined to be the least positive integer such that $u_l! \geq u_l o(\overline{H}'_{l-1})$ where $\overline{H}'_{l-1}$ is defined in (2.2). Then for $l \geq i \geq 2$, $o(H_l) \geq u_l \cdots u_i o(H_{i-1})$ where $u_l > \cdots > u_i \geq i+1$ and where H_l and H_{i-1} are defined in (2.1). Moreover, $o(H_2) \geq o(\overline{H}_2) \geq 3!|X_j|$.*

PROOF. First it is shown that $o(H_{l+1}) \geq u_{l+1}o(H_l)$ for $1 \leq l \leq t-1$. By (L4 C1), $o(\overline{H}_{l+1})|[\overline{H}_{l+1} : \overline{H}'_l]!$. Hence $[\overline{H}_{l+1} : \overline{H}'_l]! \geq [\overline{H}_{l+1} : \overline{H}_l]o(\overline{H}'_l)$. Hence $[\overline{H}_{l+1} : \overline{H}'_l] \geq u_{l+1}$. By the correspondence theorem $[\overline{H}_{l+1} : \overline{H}_l] = [H_{l+1} : H'_l] \geq u_{l+1}$. Since $H'_l > H_l$, $[H_{l+1} : H_l] \geq [H_{l+1} : H'_l]$. Hence $o(H_{l+1}) \geq u_{l+1}o(H_l)$.

Next it is shown that $u_{l+1} > u_l$ for $2 \leq l \leq t-1$. Suppose $u_{l+1} \leq u_l$. By definition $u_{l+1}! \geq u_{l+1}o(\overline{H}'_l)$. Hence $u_l! \geq u_l o(\overline{H}'_l)$ and $(u_l - 1)! \geq o(\overline{H}'_l)$. Since $[\overline{H}_l : \overline{H}'_{l-1}] \geq u_l$, $o(\overline{H}_l) \geq u_l o(\overline{H}'_{l-1})$.

$$o(\overline{H}'_l) = \frac{o(H_l)o(N_{l+1})}{o(H_l \cap N_{l+1})o(N_{l+1})} = \frac{o(H_l)}{o(H_l \cap N_{l+1})}$$

and

$$o(\overline{H}_l) = \frac{o(H_l)}{o(H_l \cap N_l)}. \tag{2.3}$$

Now $H_l \cap N_l \supset H_l \cap N_{l+1}$ since if $g \in H_l \cap N_{l+1}$ it fixes all points O_l and since it is in H_l it is in N_l. Hence $o(\overline{H}'_l) \geq o(\overline{H}_l)$. But then, $(u_{l-1})! \geq o(\overline{H}_l)$, whence $(u_{l-1})! \geq u_l o(\overline{H}'_{l-1})$, a contradiction of the choice of u_l.

Finally, it is shown that $o(H_2) \geq 6|X_j|$. Note since $o(\overline{H}'_1) \geq 2$, $u_2 \geq 3$. If $o(H_1) \geq 2|X_j|$ then, since $u_2 \geq 3$, $o(H_2) \geq 3!|X_j|$. Otherwise, by (L3), $o(H_1) = 2^a = |X_j| + 1$ for some $a \geq 1$.

Suppose $a = 1$. In this case $|X_j| = 1$. Since $o(H_1) = 2$ and $u_2 \geq 3$, $o(H_2) \geq 6|X_j|$.

Suppose $a \geq 4$. In this case, $|X_j| \geq 15$. Since H_1 is regular $H_1 \cap N_2 = \langle e\rangle$ so $o(\overline{H}'_1) = o(H_1)/o(H_1 \cap N_2) = o(H_1)$. Hence $16|o(\overline{H}'_1)$ and hence $[\overline{H}_2 : \overline{H}'_1] \geq 7$ by (L4 C1). Thus $o(H_2) \geq 6|X_j|$.

Suppose $a = 2$. In this case, $|X_j| = 3$ and $o(H_1) = 4$. Since H_1 is regular $o(H_1) = o(\overline{H}'_1) = 4$. By (L4 C1), $o(\overline{H}_2)|[\overline{H}_2 : \overline{H}'_1]!$. Hence $[\overline{H}_2 : \overline{H}'_1] \geq 5$ since $4^2 \nmid 4!$. Thus $o(H_2) \geq 5 \cdot 4 \geq 6|X_j|$.

Suppose $a = 3$. In this case, $|X_j| = 7$ and $o(H_1) = 8$. Since H_1 is regular $o(\overline{H}'_1) = 8$. If $[\overline{H}_2 : \overline{H}'_1] > 6$ the result follows. Hence $[\overline{H}_2 : \overline{H}'_1] = 5$. Hence $o(\overline{H}_2) = 40$.

Suppose $j = 1$. Then $|X_2| \leq 4$ for otherwise using (L1), $o(H_2) \geq (|X_2| + 1)o(H_1) \geq 6 \cdot 8 \geq 3!|X_1|$. Suppose $|X_2| = 3$ or 4. Since $|O_2|$ must divide 40 and $|O_2| \geq 11$, $|O_2| = 20$ or 40. Since X_2 is stabilized setwise by $\overline{H}'_1$, by (L5), $o((\overline{H}_2)_{x_2}) \geq 4$, a contradiction of $|O_2| = 20$ or 40 by (P7.4). If $|X_2| = 2$ or 1 then by (L5), $o((\overline{H}_2)_{x_2}) \geq 8$, a contradiction to $|O_2| \geq 10$.

Suppose $j > 1$. Let x_{j-1} be the point of X_{j-1} in the reducing orbit of $\bar{H}_1'$. Suppose $|X_{j-1}| \neq 1$. Note $X_{j-1} - \{x_{j-1}\}$ is stabilized by $\bar{H}_1'$ and $\alpha_{x_{j-1}}$. If $\bar{H}$ is generated by $\bar{H}_1'$ and $\alpha_{x_{j-1}}$, then $o(\bar{H}) \geqslant 40$ by (L4 C1), a contradiction to $o(\bar{H}) < o(\bar{H}_2) = 40$. Let $|X_{j-1}| = 1$. Then $|O_2|\,|40$. Since $|X_j \cup X_{j-1}| = 8$, $|X_{j-2} \cap O_2| = 2$, 12, or 32. Suppose $|O_2 \cap X_{j-2}| = 2$. Since $3 \nmid o(\bar{H}_2)$, $\alpha_{x_{j-1}}$ has cycles $(x_{j-2})(x'_{j-2}x_{j-1})$. Since $X_{j-2} \cap O_2$ is stabilized by $\bar{H}_1'$, $o((\bar{H}_1')_{x_{j-2}}) \geqslant 4$. Hence $o((\bar{H}_2)_{x_{j-2}}) \geqslant 8$, a contradiction to $o((\bar{H}_2)_{x_{j-2}}) = o(\bar{H}_2)/|O_2| = 4$. Let $|X_{j-2} \cap O_2| = 12$. Since C_5 is a normal subgroup of $\bar{H}_2$ the orbits of C_5 are a complete block system of $\bar{H}_2$. No point of X_j in a block with x_{j-1} can be reduced since the reducing maps of X_j do not have order 5. Hence the block containing x_{j-1} contains 4 points of X_{j-2}. However, if $(x_j)\alpha_{x_j} = x_{j-1}$, x_j is in a block containing 4 distinct points of X_{j-2}. Since this can be true for only two x_j, a contradiction follows. Let $|X_{j-2} \cap O_2| = 32$. Then $\bar{H}_2$ is regular on O_2 and hence by (L3 C1) cannot reduce $O_2 \cap (X_j \cup X_{j-1})$. Hence (L6) is proved.

COROLLARY 6.1. *Let A act on a set X. Let $X = \bigcup_{i=0}^{t} X_i$ be a reducible partition. Then $o(A) \geqslant (t+1)!$ and $o(A) = (t+1)!$ only if there is a reducing orbit O of A such that $|O| = t+1$ and $A^0 \cong S_{t+1}$.*

PROOF. Let O be a reducing orbit of A such that $|O \cap X_t| \geqslant 1$. Let $H_t = \langle \alpha_z$: $z \in O \cap \bigcup_{i \geqslant 1} X_i\rangle$. If $t = 1$ then $o(A) \geqslant 2$ and $o(A) = 2$ implies $A^0 \cong C_2$ and the result is clear. If $t > 1$ then by (L6) for every $j \geqslant 1$, $o(H_t) \geqslant (t+1)!|X_j|$. Hence $o(A) \geqslant (t+1)!$ and each X_j is a singleton for $j \geqslant 1$ if $o(A) = (t+1)!$. Suppose $o(A) = (t+1)!$. Let $H_{t-1} = \langle \alpha_z: z \in O \cap \bigcup_{i=1}^{t-1} X_i\rangle$. Then $o(H_{t-1}) \geqslant t!$ by (L6). But H_{t-1} fixes X_t. Thus $o(A^0) \leqslant |O|t!$ and hence $|O| = t+1$. Hence $A^0 \cong S_{t+1}$.

COROLLARY 6.2. *Let A act on a set X. Let $X = \bigcup_{i=0}^{t} X_i$ be a reducible partition. Let $H_1 = \langle \alpha_z: z \in X_j\rangle$. Suppose either*

$$H_2 = \langle \alpha_z: z \in X_j \cup X_{j+1}\rangle \quad \textit{and} \quad H_3 = \langle \alpha_z: z \in X_j \cup X_{j+1} \cup X_{j+2}\rangle \tag{2.4}$$

and there is a reducing orbit O_2 of H_2 meeting X_j in at least 2 points, or

$$H_2 = \langle \alpha_z: z \in X_{j-1} \cup X_j\rangle \quad \textit{and} \quad H_3 = \langle \alpha_z: z \in X_{j-2} \cup X_{j-1} \cup X_j\rangle \tag{2.5}$$

and $|X_j| > 2$.

Then $o(H_3) \geqslant o(\bar{H}_3) \geqslant 120$ where $\bar{H}_3$ is defined as in (2.2).

PROOF. Let H_2 and H_3 be as in (2.4). Let $\bar{H}_2$ and $\bar{H}_1'$ be as in (2.2) acting on the orbit O_2. Since $|X_j \cap O_2| \geqslant 2$, $o(\bar{H}_1') \geqslant 4$ by (L3). Suppose $o(\bar{H}_1) = 4$. By (L4 C1), $[\bar{H}_2 : \bar{H}_1'] \geqslant 5$. Also $o(\bar{H}_1') \neq 5$ since C_5 has only a single reducing map. Suppose $o(\bar{H}_1) = 6$; then $[\bar{H}_2 : \bar{H}_1'] \geqslant 4$ by (L4 C1). Suppose $o(\bar{H}_1) \geqslant 7$; then $[\bar{H}_2 : \bar{H}_1] \geqslant 5$. Using the calculation of (2.3), $o(H_2') \geqslant o(\bar{H}_2)$. Suppose $o(\bar{H}_2') \geqslant 24$; then by (L4 C1), $[\bar{H}_3 : \bar{H}_2'] \geqslant 5$ and $o(\bar{H}_3) \geqslant 120$.

Let H_3 and H_2 be as in (2.5). If there is a reducing orbit of H_2 meeting X_j in at least 2 points the argument follows as above. Suppose not. Let $O_{21}, \ldots, O_{2k}$ be the reducing orbits of H_2. Let N_2 be the pointwise stabilizer of $\bigcup_{i=1}^{k} O_{2i}$. Let $\bar{H}_2 = H_2/N_2$. Let $H_1' = \langle H, N_2\rangle$ and let $\bar{H}_1' = H_1'/N_2$. By (L3), $o(\bar{H}_1') \geqslant 4$. This case now follows to the point of bounding $o(\bar{H}_2)$ as above using (L4 C2) in place of (L4 C1). Let $O_{31}, \ldots, O_{3k'}$ be the reducing orbits of H_3. Let N_3 be the pointwise stabilizer of $\bigcup_{i=1}^{k'} O_{3i}$. Let $\overline{H}_3 = H_3/N_3$. Let $H_2' = \langle H_2, N_3 \rangle$ and let $\overline{H}_2' = H_2'/N_3$. The case concludes as above using (L4 C2) in place of (L4 C1).

Let A act on a set X. Let $X = \bigcup_{i=0}^{t} X_i$ be a reducible partition. For $1 \leq l \leq t$ let

$$H_l = \langle \alpha_z : z \in X_{t-(l-1)} \cup \cdots \cup X_t \rangle. \tag{2.6}$$

For $2 \leq l \leq t$ let $O_{l1}, \ldots, O_{lk_l}$ be the reducing orbits of H_l. Let N_l be the pointwise stabilizer $\bigcup_{v=1}^{k_l} O_{lv}$. For $2 \leq l \leq t$, let

$$\begin{aligned} \overline{H}_l &= H_l/N_l \quad \text{and for } 1 \leq l \leq t-1 \text{ let} \\ H_l' &= \langle H_l, N_{l+1} \rangle \quad \text{and let} \\ \overline{H}_l' &= H_l'/N_{l+1}. \end{aligned} \tag{2.7}$$

COROLLARY 6.3. *Let A act on a set X. Let $X = \bigcup_{i=0}^{t} X_i$ be a reducible partition. Suppose $2 \leq l \leq t$ and H_1 has at least two reducing orbits. For $2 \leq l \leq t$, u_l is defined to be the least positive integer such that $u_l! \geq u_l o(\overline{H}_{l-1}')$ where $\overline{H}_{l-1}'$ is defined in* (2.7). *Then for $2 \leq i \leq l$, $o(H_l) \geq u_l \cdots u_i o(H_{i-1})$ where $u_l > \cdots > u_i \geq i + 1$ and where H_l and H_{i-1} are defined in* (2.6). *Moreover $u_2! \geq u_2 o(\overline{H}_1')$.*

PROOF. By (L4 C2), $[\overline{H}_l : \overline{H}_{l-1}'] \geq u_l$ for $2 \leq l \leq t$ where $\overline{H}_l$ and $\overline{H}_{l-1}'$ are defined. By the correspondence theorem $[\overline{H}_l : H_{l-1}'] \geq u_l$. Since $o(H_{l-1}') \geq o(H_{l-1})$, $o(H_l) \geq u_l o(H_{l-1})$. Using the calculation in (2.3), it can be shown as in (L6) that for $3 \leq l \leq t$, $u_l > u_{l-1}$ or the defining property of u_{l-1} is contradicted. Hence for $2 \leq l \leq t$ and $l \geq i \geq 2$, $o(H_l) \geq u_l \cdots u_i o(H_{i-1})$ and $u_l > \cdots > u_2 \geq 3$. Since $\overline{H}_1'$ acts on several reducing orbits, $o(\overline{H}_1') \geq 2|X_t|$ by (L3). Hence $u_2! \geq u_2 \cdot 2 \cdot |X_t|$.

COROLLARY 6.4. *Let A act on a set X. Let $X = \bigcup_{i=0}^{t} X_i$ be a reducible partition. Suppose $t \geq 3$ and $|X_t| \geq 2$. Let H_l and $\overline{H}_l$ be defined as in* (2.6) *and* (2.7) *for $1 \leq l \leq t$. Then for $3 \leq l \leq t$, $o(H_l) \geq o(\overline{H}_l) \geq (l + 2)!$.*

PROOF. By (L6 C2), $o(\overline{H}_3) \geq 5!$ if $|X_t| \geq 3$. Suppose it has been shown for $3 \leq l \leq t - 1$ that $o(\overline{H}_l) \geq (l + 2)!$. By the calculation of (2.3), $o(\overline{H}_l') \geq o(\overline{H}_l)$. By the definition of u_{l+1}, $u_{l+1} \geq l + 3$ since $o(\overline{H}_l') \geq (l + 2)!$. Hence $o(H_{l+1}) \geq o(\overline{H}_{l+1}) \geq (l + 3)o(H_l) \geq (l + 3)!$.

LEMMA 7. *Let A be a group acting on a set X. Let $\bigcup_{i=0}^{s} X_i$ be a reducible partition of X. Let $H \subseteq A$. Suppose Y is an orbit of H where $Y \subset X_j$, $j \geq 1$. Suppose $Z \subseteq X_{j+1}$ such that $I(Z) \subset Y$. Let $K = \langle H_1, \langle \alpha_z : z \in Z \rangle \rangle$. Suppose that if O is the orbit of K containing Y, then $O \cap X_j = Y$. Let $L = \langle K, \langle \alpha_y : y \in Y \rangle \rangle$. Suppose that if O' is the orbit of L containing Y then $O' \cap X_j = Y$. Let*

$$m = \min\{|Y| - 1, |Z|\}.$$

It is the case that $o(K_y) \geq (m + 1)o(H_y)$, for $y \in Y$. Moreover, $o(L_y) \geq |Y|o(K_y)$.

PROOF. The following propositions are used in the proof.

PROPOSITION 7.1. *Let A be a group acting on a set X. Let $X = \bigcup_{i=0}^{s} X_i$ be a reducible partition. Suppose that $H \subseteq A$ and that O is an orbit of H. Let $x \in O$ and let $(x)\psi_x \notin O$. Then if $h \in H$, $h\psi_x^{-1} \notin H$.*

PROOF. Since $x \in O$ and $(x)\psi_x \notin O$, $\psi_x \notin H$. Hence $h\psi_x^{-1} \notin H$.

PROPOSITION 7.2. *Let A be a group acting on a set X. Let $X = \bigcup_{i=0}^{s} X_i$ be a reducible partition. Suppose that $H \subseteq A$ and that O is an orbit of H. Suppose $\{z_r$: $1 \leqslant r \leqslant k\} \subset O$ is such that $(z_r)\alpha_{z_r} \notin O$ and $(z_s)\alpha_{z_r} \in O$ if $r \neq s$ for $1 \leqslant r \leqslant k$ and $1 \leqslant s \leqslant k$. If $h \in H$, α_{z_s} and α_{z_r} are such that $r \neq s$, then $h\alpha_{z_s}^{-1} \notin H$.*

PROOF. Since $(z_s)\alpha_{z_s} \in O$ and O is an orbit of H, $(z_s)\alpha_{z_s}h^{-1} \notin O$. Now $((z)\alpha_{z_s}h^{-1})h\alpha_{z_s}\alpha_{z_r} = (z_s)\alpha_{z_r} \in O$ by assumption. Since O is an orbit of H, $h\alpha_{z_s}^{-1}\alpha_{z_r} \notin H$.

PROPOSITION 7.3. *Let A be a group acting on a set X. Let $X = \bigcup_{i=0}^{s} X_i$ be a reducible partition. Let $H \subseteq A$. Let Y be part of an orbit of H. Suppose $\alpha \notin H$ and $(y)\alpha \in Y$. Then there exists $h \in H$ such that $h\alpha^{-1} \in K_y$.*

PROOF. Since Y is part of an orbit of H there exists an h such that $(y)h = (y)\alpha_x$. Hence $h\alpha_x^{-1} \in K_y$.

PROPOSITION 7.4. *Let A be a group acting on a set X. Suppose $H \subseteq A$ and O is an orbit of H. Then for every $x \in O$, $o(H_x) = o(H)/|O|$.*

PROOF. This is a basic fact of permutation group theory.

The main argument of the proof begins.

Since $|Z| \geqslant m$ there exists a set of distinct points $\{z_1, \ldots, z_m\}$ with $\{\alpha_{z_1}, \ldots, \alpha_{z_m}\}$ the set of corresponding reducing maps. Since $O \cap X_j = Y$, $(Y - (z_k)\alpha_{z_k}^{-1})\alpha_{z_k} \subset Y$, for any $1 \leqslant k \leqslant m$. Hence at most m distinct points of Y are mapped outside Y by one or more of the α_{z_i}, $1 \leqslant i \leqslant m$. Hence since $|Y| - 1 \geqslant m$, there exists a $y \in Y$ such that $(y)\alpha_{z_i} \in Y$, for $1 \leqslant i \leqslant m$. By (P7.3) there exists an $h_i \in H$ for $1 \leqslant i \leqslant k$ such that $h_i\alpha_{z_i}^{-1} \in K_y$. By (P7.1), $h_i\alpha_{z_i}^{-1} \notin H_y$, for $1 \leqslant i \leqslant k$. Let h and h' be two elements of H_y. If $hh_i\alpha_{z_i}^{-1} = h'h_k\alpha_{z_k}^{-1}$, $i \neq k$, then $hh_i = h'h_k\alpha_{z_k}^{-1}\alpha_{z_i}$. By (P7.2), $h'h\alpha_{z_k}^{-1} \notin H$, a contradiction. Thus for every map of H_y there are $m + 1$ maps of K_y, namely $h, h\alpha_{z_1}^{-1}, \ldots, h\alpha_{z_m}^{-1}$. The maps arising from different elements of H_y are clearly distinct. Hence $o(K_y) \geqslant (m + 1)o(H_y)$.

If $|Y| = 1$, then $o(L_y) \geqslant o(K_y)$ since $L \supset K$. Suppose $|Y| > 1$. Fix $y^- \in Y$. Let $y \in Y - y^-$. By (P7.3) there exists $k_y \in K$ such that $k_y\alpha_y^{-1} \in L_{y^-}$. By (P7.1) $k_y\alpha_y^{-1} \notin K_{y^-}$. There are $|Y| - 1$ such maps. Suppose k and k' are elements of K_y. Suppose $kk_{y'}\alpha_{z_{y'}}^{-1} = k'k_y\alpha_{z_y}^{-1}$ where y and $y' \in Y - \{y^-\}$. Then $kk_{y'} = k'k_y\alpha_{z_y}^{-1}\alpha_{z_{y'}}$, a contradiction of (P7.2). Hence for every map of K_y there are $|Y|$ maps of L_y. Hence $o(L_y) \geqslant |Y|o(K_y)$.

COROLLARY 7.1. *Let A be a group acting on a set X. Let $X = \bigcup_{i=0}^{s} X_i$ be a reducible partition of X. Let $H \subseteq A$. Suppose $Y = O \cap X_j$, for some $j \geqslant 1$ and for some orbit O of H. Suppose Z is contained within an orbit O_Z of H distinct from O and disjoint from X_0. Suppose for all distinct $z, z' \in Z$ that $(z)\alpha_{z'} \in Z$. Suppose $I(Z) \cap O_Z = \varnothing$. Let $K = \langle H, \langle \alpha_z\colon z \in Z \rangle \rangle$. Let O' be the orbit of K containing O. Suppose $O' \cap X_1 = Y$. Suppose $\{(z)\alpha_z^{-1}\colon z \in Z\} \cap Y = \varnothing$. Then $o(K_y) \geqslant (|Z| + 1)o(H_y)$.*

PROOF. Let $y \in Y$. Let $z \in Z$. Suppose $(y)\alpha_z \notin Y$. Then since $O' \cap X_j = Y$, $(y)\alpha_z = z$. However, this contradicts the hypothesis that $\{(z)\alpha_z^{-1}: z \in Z\} \cap Y = \varnothing$. Hence $(y)\alpha_z \in Y$. By (P7.3) there exists an $h \in H$ such that $h\alpha_z^{-1} \in K_y$. Then $(z)\alpha_z \notin O_Z$, since $I(Z) \cap O_Z = \varnothing$. Hence by (P7.1), $h\alpha_z^{-1} \notin H_y$. Let h and $h' \in H_y$. Suppose $hh_y\alpha_z^{-1} = h'h_{y}\text{-}\alpha_{z'}^{-1}$ for $z \neq z'$. Then $hh_y = h'h_y\text{-}\alpha_{z'}^{-1}\alpha_z$. This contradicts (P7.2). Thus for every element $h \in H_y$ there are $|Z| + 1$ elements of K_y, namely $\{h\} \cup \{h\alpha_z^{-1}: z \in Z\}$. Hence $o(K_y) \geqslant (|Z| + 1)o(H_y)$.

LEMMA 8. *Let A be a group acting on a set X. Let $X = \bigcup_{i=0}^{s} X_i$ be a reducible partition. Suppose $Y = X_i$, $i \geqslant 1$, breaks into at least two orbits under H. Let O be an orbit of H on Y. Suppose $|O| \geqslant 2$. Then if $y \in O$ and $K = \langle H, \langle \alpha_y: y \in Y\rangle\rangle$, then $o(K_y) > o(H_y)$.*

PROOF. Let $y \in O$. Suppose there exists a $y^* \in Y - y$ such that $(y)\alpha_{y^*} \in O$. By (P7.3) there exists an $h \in H$ such that $h\alpha_{y^*}^{-1} \in K_y$. Since $(y^*)\alpha_{y^*} \notin O$ by (P7.1), $h\alpha_{y^*}^{-1} \notin H_y$. Hence $o(K_y) > o(H_y)$. Suppose for all $y^* \in Y - y$, $(y)\alpha_{y^*} \in O$. Since $|Y| - 1 > |Y| - |O|$ by the pigeon-hole principle some y^- is the image of y under $\alpha_{y'}^{-1}$ and $\alpha_{y''}^{-1}$. Hence $\alpha_{y'}^{-1}\alpha_{y''} \in K_k \setminus H_y$. Hence $o(K_y) > o(H_y)$.

Let A be a group acting on a set X. Suppose that $X = \bigcup_{i=0}^{s} X_i$ is a reducible partition of X. Suppose $Y \subset X_j$ for some $j \geqslant 1$. Suppose $H \subseteq A$ and suppose $O_1, \ldots, O_u$ are the orbits of H where $Y \subset \bigcup_{k=1}^{u} O_k$. Suppose $I(Y) \cap \bigcup_{k=1}^{u} O_k = \varnothing$. Suppose for every $y \in Y$ and $a \in \bigcup_{k=1}^{u} O_k$ that $(a)\alpha_y \notin U_y$ where U_y is the orbit of H containing $(y)\alpha_y$. Let

$$K = \langle H, \langle \alpha_z: z \in Y\rangle\rangle. \tag{2.8}$$

Let $Q_1, \ldots, Q_{u'}$ be the orbits of K. Let M be the pointwise stabilizer of $\bigcup_{m=1}^{u'} Q_m$. Let

$$\begin{aligned} K^- &= K/M, \\ H' &= \langle H, M\rangle, \quad \text{and} \\ H^- &= H'/M. \end{aligned} \tag{2.9}$$

LEMMA 9. *Let K and H be as in* (2.8). *Let K^- and H^- be as in* (2.9). *If f is the least integer such that $fo(H^-) \leqslant f!$ then $[K^- : H^-] \geqslant f$ and $[K : H] \geqslant f$.*

PROOF. The proof follows the proof of (L4 C2). Suppose H^- contains a nontrivial normal subgroup N^- of K^-. Since N^- is nontrivial it acts nontrivially on one of the orbits Q_m, $1 \leqslant m \leqslant u'$. However, the orbits of N are blocks of K on Q_m. By assumption, there exists a k such that $Q_m \supset O_k$. Hence $Q_m \cap Y \neq \varnothing$. Since N has nontrivial orbits on Q_m there exists a nontrivial orbit O of N containing $y \in O_s \cap Y$. Hence there exists an element $n \in N$ containing a cycle $(ya \cdots)$ where $a \in O_k$. The cycle $((y)\alpha_y(a)\alpha_y \cdots)$ is contained by $\alpha_y^{-1}n\alpha_y$. By definition $(y)\alpha_y \in U$ and $(a)\alpha_y \notin U_y$. Since $\alpha_y^{-1}n\alpha_y \in N \subset H^-$ a contradiction follows. As in (L4 C1) H^- containing no nontrivial normal subgroup of K^- means that $o(K^-)|[K^- : H^-]!$. Hence $[K^- : H^-] \geqslant f$. By the correspondence theorem $[K : H'] \geqslant f$.

Let A be a group acting on a set X. Let $X = \bigcup_{i=0}^{s} X_i$ be a reducible partition. Let $H = \langle \alpha_z : z \in \bigcup_{i=r}^{s} X_i \rangle$. Suppose H has reducing orbits $O_1, \ldots, O_p$ such that $X_s \cap O_v \neq \emptyset$ for $1 \leqslant v \leqslant p$. For $1 \leqslant u \leqslant p$ and $1 \leqslant k \leqslant r - s + 1$ let

$$H_{uk} = \left\langle \alpha_z : z \in \left(\bigcup_{v=1}^{u-1} \bigcup_{m=r}^{s} O_v \cap X_m \right) \cup \left(\bigcup_{m=s-(k-1)}^{s} O_u \cap X_m \right) \right\rangle. \quad (2.10)$$

If $k > 1$ then let H_{uk-1} have reducing orbits $O_1^-, \ldots, O_c^-$ with $\bigcup_{w=1}^{c} O_w^- \subseteq O_u$. Let H_{uk} act on reducing orbits $Q_1, \ldots, Q_d$ with $\bigcup_{e=1}^{d} Q_e \subseteq O_u$. Let N_{uk} be the pointwise stabilizer of $\bigcup_{e=1}^{d} Q_e$. Let

$$\begin{aligned} H_{uk}^- &= H_{uk}/N_{uk}, \\ H'_{uk-1} &= \langle H_{uk-1}, N_{uk} \rangle, \quad \text{and} \\ H_{uk-1}^{-\prime} &= H'_{uk-1}/N_{uk}. \end{aligned} \quad (2.11)$$

COROLLARY 9.1. *Let A be a group acting on a set X. Let $X = \bigcup_{i=0}^{s} X_i$ be a reducible partition. For $1 \leqslant u \leqslant p$ and $1 \leqslant k \leqslant r - s + 1$ let H_{uk} be defined in* (2.10). *If $u = 1$ and $k = 1$ then $o(H_{11}) \geqslant (|X_s \cap O_1| + 1)$. If $u > 1$ and $k = 1$ then $o(H_{u1}) \geqslant (|X_s \cap O_s| + 1)o(H_{u-1s-r+1})$. Let H_{uk}^- and $H_{uk}^{-\prime}$ be defined in* (2.11). *Let f_{uk} be the least integer such that $f_{uk}! \geqslant f_{uk}o(H_{uk-1}^{-\prime})$, where for $3 \leqslant k \leqslant s - r + 1$ and $1 \leqslant u \leqslant p$, $f_{uk} > f_{uk-1}$ and $f_{u2} \geqslant 3$. Then* (1) *$[H_{uk}^- : H_{uk-1}^{-\prime}] \geqslant f_{uk}$, $[H_{uk} : H_{uk-1}] \geqslant f_{uk}$, and $o(H_{u2}^-) \geqslant 6|X_s \cap O_u|$.* (2) *$o(H_{uk-1}) \geqslant f_{um'} \cdots f_{un}o(H_{un-1})$ where $1 \leqslant u \leqslant p$ and $2 \leqslant n \leqslant m' \leqslant s - r + 1$.* (3) *$o(H) \geqslant \prod_{v=1}^{p} (\prod_{k=2}^{s-r+1} f_{vk}(|X_s \cap O_v| + 1))$.*

PROOF. Suppose $u = 1$ and $k = 1$. Then $o(H_{11}) \geqslant |X_s \cap O_1| + 1$ by (L3). Suppose $u > 1$ and $k = 1$. By (L1), $o(H_{u1}) \geqslant (|X_s \cap O_u| + 1)o(H_{u-1s-r+1})$. Suppose $k > 1$. Then H_{uk}^- and $H_{uk-1}^{-\prime}$ satisfy the hypothesis of (L9). Hence $[H_{uk}^- : H_{uk-1}^{-\prime}] \geqslant f_{uk}$ and by the correspondence theorem $[H_{uk} : H_{uk-1}] \geqslant f_{uk}$.

Suppose for some $3 \leqslant k \leqslant s - r + 1$ and $1 \leqslant u \leqslant p$, $f_{uk} \leqslant f_{uk-1}$. By definition $o(H_{uk-1}^-) \geqslant f_{uk-1}o(H_{uk-2}^{-\prime})$. Hence $(f_{uk-1} - 1)! \geqslant o(H_{uk-1}^{-\prime})$ since $f_{uk}o(H_{uk-1}^{-\prime}) \leqslant f_{uk}!$ and $f_{uk} \leqslant f_{uk-1}$. To obtain a contradiction the proof proceeds as in (L6) with the following elaboration of the calculation made in (2.3). It is necessary to show that $H_{uk-1} \cap N_{uk} \subseteq H_{uk-1} \cap N_{uk-1}$. If $\varphi \in H_{uk-1} \cap N_{uk}$ then φ fixes every point of $\bigcup_{e=1}^{d} Q_e$. However $\bigcup_{e=1}^{d} Q_e \supset \bigcup_{w=1}^{c} O_w^-$. Since $\varphi \in H_{uk-1}$ and φ fixes every point of $\bigcup_{w=1}^{c} O_w^-$, $\varphi \in H_{uk-1} \cap N_{uk-1}$ and the result holds.

Since $o(H_{u1}^{-\prime}) \geqslant 2$, $f_{u2} \geqslant 3$. Repeated application of $o(H_{uk}) \geqslant f_{uk}o(H_{uk-1})$ gives $o(H_{um'}) \geqslant f_{um'} \cdots f_{un}o(H_{un-1})$ where $1 \leqslant u \leqslant p$ and $2 \leqslant n < m' \leqslant s - r + 1$. By the argument of (L6), $o(H_{u2}^-) \geqslant 6|X_s \cap O_u|$ for $1 \leqslant u \leqslant p$. In summary $o(H) \geqslant \prod_{v=1}^{p}(\prod_{k=2}^{s-r+1} f_{vk}(|X_s \cap O_v| + 1))$.

Let A be a group acting on a set X. Let $X = \bigcup_{i=0}^{s} X_i$ be a reducible partition of X. Let $H \subseteq A$. Let O_1 and O_2 be two orbits of H such that $O_1 \subset \bigcup_{i=1}^{r+1} X_i$ where $0 \leqslant r \leqslant s$, and $O_2 \subset \bigcup_{i=r+1}^{s} X_i$ where $O_2 \cap X_{r+1} \neq \emptyset$. Suppose there exists a $z \in O_2 \cap x_{r+1}$ such that $(z)\alpha_z \in O_1$. Let $K_0 = H$ and let $Q_0 = O$. For $1 \leqslant j \leqslant v$ let $K_i = \langle K_{i-1}, \alpha_{z_i} \rangle$ where $z_i \in Q_{i-1} \cap X_{r+1}$ and $\alpha_{z_i} \notin K_i$ and let Q_i be the orbit of K_i containing Q_{i-1}. Since A is finite, u is bounded. Let $K = K_u$ and $Q = Q_u$. K is termed an *O_1 extension of H*.

COROLLARY 9.2. *Suppose A is a group acting on a set X. Suppose $X = \bigcup_{i=0}^{s} X_i$ is a reducible partition. Let $H \subseteq A$ with orbits $O_1 \subset \bigcup_{i=1}^{r+1} X_i$ where $0 \leq r < s$ and $O_2 \subset \bigcup_{i=r+1}^{s} X_i$ with $O_2 \cap X_{r+1} \neq \emptyset$. Let K be an O_1 extension of H. Let $N = K_Q$ and $H' = \langle H, N \rangle$. Let $K^- = K/N$ and $H^{-\prime} = H'/N$. Then if a is the least integer such that $a! \geq ao(H^{-\prime})$ then $[K : H] \geq a$.*

PROOF. This is a direct consequence of (L4 C1).

3. A lower bound for $o(A_x)$.

THEOREM 1. *Let A be a group acting on a set X. Let $X = \bigcup_{i=0}^{t} X_i$ be a reducible partition of X. Then there exists an $x \in X$ such that $o(A_x) \geq t!$.*

PROOF. The case $t = 1$ is clear. Let $t = 2$. Let $H = \langle \alpha_z : z \in X_2 \cup X_1 \rangle$. Suppose H acts on a single orbit $O = X_0 \cup X_1 \cup X_2$. By (L3 C1), H is not regular. Hence for $x \in O$, $o(A_x) \geq 2$.

Let $t \geq 3$. *Suppose $\{X_i\}_{i=1}^{t-1}$ are singletons.* Let $u = |X_t|$. Let $H_j = \langle \alpha_z : z \in X_t \cup \cdots \cup X_{t-(j-1)} \rangle$, $1 \leq j \leq t-1$. By (L3), $o(H_1) \geq u + 1$. Since H_j fixes $X_{t-(j+1)}$, $o(H_{j+1}) \geq (j + 1 + u)o(H_j)$, $1 \leq i \leq t-2$. Hence

$$o(H_{t-1}) \geq (t - 1 + u)o(H_{t-2}).$$

Hence $o(H_{t-1}) \geq (t - 1 + u) \cdots (u + 2)(u + 1)$. Let $x_1 \in X_1$. Since $\bigcup_{i=1}^{t} X_i$ is an orbit of H_{t-1} and $|\bigcup_{i=1}^{t} X_i| = u + t - 1$, by (P7.4),

$$o(A_{x_1}) \geq o(H_{t-1})/(t + u - 1) \geq (t + u - 2) \cdots (u + 1).$$

Since α_{x_1} fixes x_2 by (L7 C1), $o(A_{x_2}) \geq 2 \cdot (t + u - 2) \cdots (u + 1)2 \geq t!$ provided $u \geq 2$. If $u = 1$ let $H = \langle \alpha_z : z \in \bigcup_{i=1}^{t-1} X_i \rangle$. Since x_t is fixed by H, $o(A_{x_t}) \geq o(H)$. However, $o(H) \geq t!$ by (L6).

Suppose not all of the $\{X_i\}_{i=1}^{t-1}$ are singletons. Suppose $t \geq 4$. Let $|X_t| \leq t + 1$. Suppose $|X_{t-1}| \geq 2$. Let $H_1 = \langle \alpha_z : z \in X_{t-1} \rangle$ and $H_2 = \langle \alpha_z : z \in X_{t-1} \cup X_{t-2} \rangle$. If H_3 reduces X_{t-2}, in addition $o(H_3) \geq 120$ by (L6 C4). Let $H_{t-1} = \langle \alpha_z : z \in \bigcup_{i=1}^{t-1} X_i \rangle$. By (L6 C4), $o(H_{t-1}) \geq (t + 1)!$. Since $H_{t-2} \subseteq A_{(X_t)}$ and $|X_t| \leq t + 1$, $o(A_{x_t}) \geq t!$, where $x_t \in X_t$.

Suppose $|X_{t-1}| = 1$. *Suppose* $t \geq 5$. The arguments just applied to the case $|X_{t-1}| \geq 2$ show that if $H_{t-2} = \langle \alpha_z : z \in \bigcup_{i=1}^{t-2} X_i \rangle$ is generated reducing from the top then $o(H_{t-2}) \geq t!$. As $H_{t-2} \subseteq A_{x_{t-1}}$ this implies $o(A_{x_{t-1}}) \geq t!$. Hence suppose $|X_{t-2}| = 1$ if $t \geq 5$.

Suppose $|X_1| \geq |X_t|$. If $|X_1| = 1$ then $|X_t| = 1$ and the result follows by (L6).

Suppose $|X_1| \geq 2$. Let $H_{t-1} = \langle \alpha_z : z \in \bigcup_{i=1}^{t-1} X_i \rangle$. By (L6), $o(H_{t-1}) \geq t!|X_1|$ and the result follows using $H_{t-1} \subseteq A_{(X_t)}$.

Suppose $|X_t| > |X_1|$. By the above argument $|X_{t-1}| = 1$. Let $|X_t| = 2$. Then $H_1 \cong S_3$. If $H_{t-2} = \langle \alpha_z : z \in \bigcup_{i=3}^{t} X_i \rangle$, then by (L6 C4), $o(H_{t-2}) \geq t!$. Since $|X_1| = 1$ the result follows.

Suppose $|X_1| = 1$ *and* $|X_t| \geq 3$. By (L6 C4), $o(H_3) \geq 5!$ and so if $t \geq 5$ then $o(H_{t-2}) \geq t!$ by (L6 C4). Hence $o(A_{x_1}) \geq t!$ since $H_{t-2} \subset A_{x_1}$. If $t = 4$ then $o(H_2) \geq 20$ by (L3) and (L4 C2). Since $o((H_3)_{x_3}) = o((H_3)_{x_1}) \geq o(H_2) \geq 20$ and x_3 is fixed by α_{x_1}, $o(A_{x_3}) \geq 4!$.

Suppose $|X_t| \geq 3$ *and* $|X_1| > 1$. Let $|X_t| = 3$. If $t \geq 5$ then by (L6 C4), $o(H_{t-2}) \geq t!$. If X_1 is an orbit of H_{t-2} then $o(A_{x_1}) \geq t!$ by (L7). If $|X_1|$ is not an orbit by (P7.4), $o(A_{x_1}) \geq 2t!/|X_1|$. Note that $|X_1| \leq 2$ since $|X_t| > |X_1|$. If $t = 4$ then if $H_2 = \langle \alpha_z\colon z \in X_1 \cup X_2 \rangle$, $o(H_2) \geq 12$ by (L6). Let $H_3 = \langle \alpha_z\colon z \in X_1 \cup X_2 \cup X_4 \rangle$. H_2 fixes x_3 so $o((H_3)_{x_4}) = o((H_3)_{x_3}) \geq 12$. By (L7 C1), $o((H_4)_{x_4}) \geq 24$. Let $|X_t| = 4$, 5, or 6. By (L3), $o(H_1) \geq 2|X_t|$. By (L6 C1), $o(H_{t-2}) \geq (t+1) \cdots 5 \cdot 2|X_t|$. By (L8),

$$o(A_{x_1}) \geq 2o(H_{t-2})/|X_1| \geq 2o(H_{t-2})/\left(|X_t| - 1\right)$$
$$\geq 2(t+1) \cdots 5 \cdot 2|X_t|(|X_t| - 1) \geq t!.$$

Let $|X_t| \geq 7$. Since $t \geq 6$ as $|X_t| \leq t + 1$, $|X_{t-2}| = 1$. By (L3), $o(H_1) \geq |X_t| + 1 \geq 8$. Since x_{t-2} is fixed by H_1, $o(H_2) \geq 9(|X_t| + 1) \geq 72$. By (L6), $o(H_{t-2}) \geq (t+1) \cdots 6 \cdot 9(|X_t| + 1)$. By (L8),

$$o(A_{x_1}) \geq 2o(H_{t-2})/|X_1| \geq 2o(H_{t-2})/|X_1|$$
$$\geq 2(t+1) \cdots 6 \cdot 9(|X_t| + 1)/|X_1| \geq t!.$$

Let $|X_t| \geq t + 2$ *and let* $|X_1| \geq \frac{1}{2}|X_t|$. Let $H_t = \langle \alpha_z\colon z \in \bigcup_{i=1}^{t-1} X_i \rangle$ be generated reducing from the bottom. By (L6), $o(H_{t-1}) \geq t!|X_1|$. Since X_t is not a singleton (L5) gives $o(A_{x_t}) \geq 2o(H_{t-1})/|X_t| \geq t!2|X_1|/|X_t| \geq t!$.

Suppose $|X_1| < \frac{1}{2}|X_t|$. Let $H_{t-2} = \langle \alpha_z\colon z \in \bigcup_{i=3}^{t} X_i \rangle$. Suppose $t > 4$. Hence $|X_t| \geq 7$ as $|X_t| \geq t + 2$. By (L3), $o(H_1) \geq 8$. By (L6 C3), $o(H_{t-2}) \geq (t+1) \cdots 5|X_t| \geq (t+1) \cdots 5 \cdot 2|X_1|$. If X_1 is a singleton, then $o(A_{x_1}) \geq o(H_{t-2}) \geq (t+1) \cdots 5 \cdot 8 \geq t!$. If not, apply (L8) to obtain $o(A_{x_1}) \geq 2o(H_{t-2})/|X_1| \geq t!$. Let $t = 4$ and let $|X_4| = 6$. By (L3), $o(H_1) \geq 12$. By (L6 C3), $o(H_2) \geq 5o(H_1)$. If $|X_4| \geq 7$, $o(H_1) \geq 8$ by (L3). By (L6 C3), $o(H_2) \geq 5o(H_1) \geq 40$. If X_1 is a singleton, the result follows. Suppose X_1 is not a singleton. Let $|X_4| = 6$. By (L8), $o(A_{x_1}) \geq 2 \cdot 5o(H_1)/|X_1| \geq 20|X_4|/|X_1| \geq 40$. Let $|X_4| = 7$. By (L6 C3), $o(H_2) \geq 40$. By (L8), $o(A_{x_1}) \geq (2 \cdot 40)/3 \geq 4!$. Let $8 \leq |X_4| \leq 14$. By (L3) and (L6 C3), $o(H_2) \geq 5 \cdot 2|X_4|$. By (L8), $o(A_{x_1}) \geq 2 \cdot 5 \cdot 2|X_4|/|X_1| \geq 40$. Let $|X_4| = 15$. By (L3) and (L6 C3), $o(H_2) \geq 7 \cdot 16$. By (L8), $o(A_{x_1}) \geq 2 \cdot 7 \cdot 2|X_4|/|X_1| \geq 24$. Let $|X_4| \geq 16$. By (L3), $o(H_1) > 24$. By (L6 C3), $o(H_2) \geq 6|X_4|$. By (L8), $o(A_{x_1}) \geq 2 \cdot 6 \cdot 2|X_1|/|X_1| = 24!$.

Let $t = 3$. The proposition has been shown in the case X_1 and X_2 are singletons. *Suppose* $|X_3| \leq 4$. Let $H_1 = \langle \alpha_z\colon z \in X_2 \rangle$ and $H_2 = \langle \alpha_z\colon z \in X_2 \cup X_1 \rangle$. If X_3 is a singleton then $o((H_2)_{x_3}) \geq 3!$ by (L6). Suppose X_3 is not a singleton. Suppose $|X_2| \geq 2$. By (L6 C2), $o(H_2) \geq 20$. By (L5), $o(A_{x_3}) \geq 2o(H_2)/|X_3| \geq 20/2 \geq 10$. Suppose X_2 is a singleton and X_1 is not one. Let $H_1 = \langle \alpha_z\colon z \in X_1 \rangle$. By (L6), $o(H_2) \geq 6|X_1|$. By (L5),

$$o(A_{x_3}) \geq 2o(H_2)/|X_3| \geq 6 \cdot 2|X_1|/4 \geq 6.$$

Let $|X_3| \geq 5$. Suppose $2|X_1| \geq |X_3|$. Let $H_2 = \langle \alpha_z\colon z \in X_1 \cup X_2 \rangle$. By (L6), $o(H_2) \geq 6|X_1|$. By assumption, $|X_3| \geq 5$. By (L5),

$$o((H_3)_{x_3}) \geq \frac{2o(H_2)}{|X_3|} \geq 6.$$

Suppose $|X_3| \geqslant 2|X_1|$. Let $H_1 = \langle \alpha_z\colon z \in X_3 \rangle$. If X_1 is a singleton, then since $o(H_1) \geqslant 8$, by (L3) the result follows. If X_1 is a doubleton then by (L8), $o(A_{x_1}) \geqslant 2o(H_1)/2 \geqslant 8$. *Suppose* $|X_1| \geqslant 3$. If X_1 splits into 3 orbits under H_1 then there exists an orbit $O \subseteq X_1$, with $3|O| \leqslant |X_1|$. Hence for $x_1 \in O$, $o((H_1)_{x_1}) \geqslant o(H_1)/|O| \geqslant (|X_3| + 1)/|O| \geqslant 2|X_1|/|O| \geqslant 6$. Suppose X_1 splits into 2 orbits. If either orbit is a singleton, then $o(A_{x_1}) \geqslant o((H_1)_{x_1}) \geqslant 8$. Suppose neither is a singleton. Suppose only one of the orbits of H_1 on X_1 is part of a reducing orbit of H_2 or both are part of different reducing orbits. Let O be an orbit such that $2|O| \leqslant |X_1|$. Either O is part of the reducing orbit of H_2 or it is stabilized by H_2. Since $|O| \geqslant 2$ and $\alpha_{x_2}^{-1}$ sends all but at most one element of O into O for $x_2 \in X_2$, there exists an $x_1 \in O$ such that $(x_1)\alpha_{x_2}^{-1} \in O$ for some $x_2 \in X_2$. By (P7.3) and (P7.1), there exists a $\psi \in H_1$ such that $\alpha_{x_2}^{-1}\psi^{-1} \notin (H_1)_{x_1}$. However

$$o((H_1)_{x_1}) \geqslant O(H_1)/|O| \geqslant (|X_3| + 1)/|O| \geqslant 2 \cdot 2 \cdot (|X_1| + 1)/|X_1| > 4.$$

Hence $o(A_{x_1}) \geqslant 8$. Suppose both orbits are part of a single reducing orbit of H. Since $o((H_1)_{x_4}) \geqslant 4$ for $x_1 \in O$, $o((H_2)_{x_3}) > 4$. Since X_3 is part of a single reducing orbit of H_2 and if $H_3 = \langle \alpha_z\colon z \in \bigcup_{i=0}^{3} X_i \rangle$ then $o((H_3)_{x_3}) \geqslant 2o((H_2)_{x_3}) > 8$ by (L7 C1).

Suppose X_1 is a single orbit under H_1. Since $o(H_1) \geqslant |X_3| + 1 \geqslant 2|X_1|$, $o((H_1)_{x_1}) \geqslant 2$. Since X_1 is not a singleton, $o((H_2)_{x_1}) \geqslant 4$ by (L7). Since X_3 is part of a single reducing orbit of H_2, $o((H_2)_{x_3}) \geqslant 4$. By (L7 C1), $o((H_3)_{x_3}) \geqslant 8$.

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